§ Local isometries and rigid motions (do Carmo §4.2)

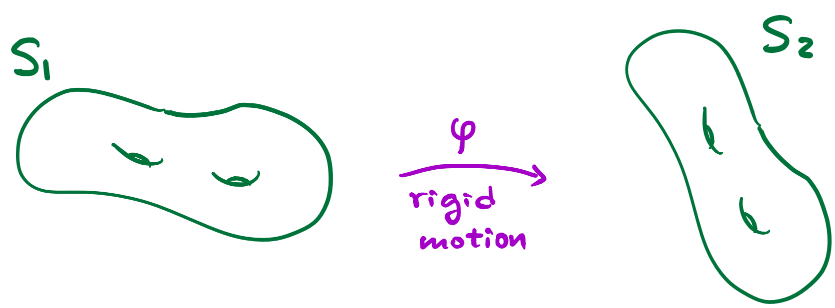
<u>Recall:</u> $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ rigid motion

$\Leftrightarrow \quad \langle d\varphi_p(v), d\varphi_p(w) \rangle = \langle v, w \rangle$

$\forall\, p \in \mathbb{R}^3, \quad \forall\, v, w \in \mathbb{R}^3 = T_p\mathbb{R}^3$

$\Leftrightarrow \quad \varphi(x) = \underbrace{A}_{\text{rotation, reflection}} x + \underbrace{b}_{\text{translation}}$ where $A \in O(3)$, $b \in \mathbb{R}^3$

<u>Def$^{\underline{n}}$:</u> Two surfaces S_1, S_2 in $\mathbb{R}^3$ are congruent if $\exists$ rigid motion $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\varphi(S_1) = S_2$.



<u>Note:</u> Congruent surfaces have the same geometry, both intrinsic and extrinsic!

intrinsic: depends only on 1^{st} f.f

extrinsic: depends both on 1^{st} & 2^{nd} f.f.

Question: Is there a transformation that **ONLY** preserves the intrinsic geometry?

Def$^{\underline{n}}$: A smooth map $\varphi : S_1 \longrightarrow S_2$ between surfaces is said to be a local isometry if $\forall p \in S_1$,

$$d\varphi_p : T_pS_1 \longrightarrow T_{\varphi(p)}S_2 \quad \text{is a linear isometry}$$

i.e. $\langle d\varphi_p(v), d\varphi_p(w) \rangle = \langle v, w \rangle \quad \forall p \in S_1$
$\forall v, w \in T_pS_1$

If, furthermore, φ is a diffeomorphism, then we say that φ is an isometry.

Example: Any rigid motion $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ restricts to an isometry $\varphi|_S : S \to \varphi(S)$.

Def$^{\underline{n}}$: (1) Two surfaces S and S' are isometric if $\exists$ isometry $\varphi : S \to S'$.

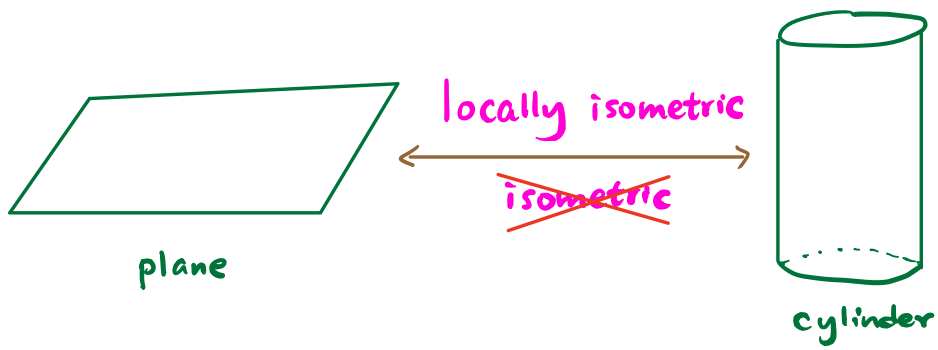
(2) Two surfaces S and S' are locally isometric if $\forall p \in S$, $\exists$ nbd $V \subseteq S$ and a local isometry

$$\varphi : V \to S'$$

and $\forall p' \in S'$, $\exists$ nbd $V' \subseteq S'$ and a local isometry

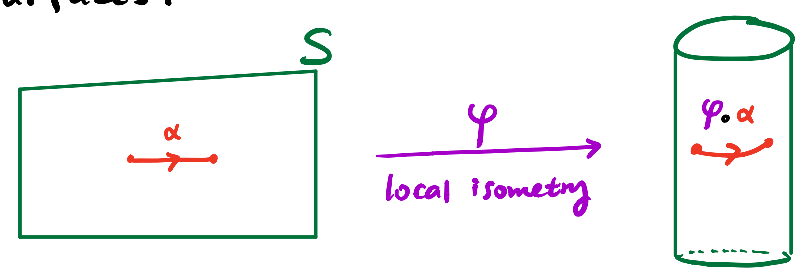
$$\varphi : V' \to S$$

Remark: Two surfaces can be locally isometric without being isometric, for example,



Basically, "intrinsic geometry" is the study of properties / quantities that are invariant under isometries!

Prop: Local isometries preserve the length of curves on surfaces.



Proof:

$$\underbrace{\int_a^b \|\alpha'(t)\|\,dt}_{\text{Length}(\alpha)} = \underbrace{\int_a^b \|d\varphi_{\alpha(t)}(\alpha'(t))\|\,dt}_{\text{Length}(\varphi\circ\alpha)}$$

$\square$

Ex: Area is preserved under isometries.

<u>Note</u>: Since a plane has $H \equiv 0$ but $H \not\equiv 0$ for cylinders, the mean curvature H is <u>NOT</u> intrinsic.

However, we will see later that the Gauss curvature K is actually intrinsic!

<u>Q</u>: How to decide if two surfaces S and $\hat{S}$ are locally isometric?

<u>A</u>: YES if they have the "same" 1st fundamental form.

<u>Prop</u>: Let $\mathbf{X} : \mathcal{U} \to S$ and $\hat{\mathbf{X}} : \mathcal{U} \to \hat{S}$ be parametrizations of the surfaces S and $\hat{S}$ from the <u>SAME</u> domain $\mathcal{U} \subseteq \mathbb{R}^2$ s.t.

$$\begin{array}{c}\text{1st f.f.} \\ \text{of } S \\ \text{w.r.t. } \mathbf{X}\end{array} = (g_{ij}) \underset{\substack{\uparrow \\ \text{as } 2\times 2 \\ \text{matrices} \\ \text{of functions} \\ \text{on } \mathcal{U}}}{=} (\hat{g}_{ij}) = \begin{array}{c}\text{1st f.f.} \\ \text{of } \hat{S} \\ \text{w.r.t. } \hat{\mathbf{X}}\end{array}$$

Then,

$$\hat{\mathbf{X}} \circ \mathbf{X}^{-1} : \underset{\substack{\text{\rotatebox{90}{$\subseteq$}} \\ S}}{\mathbf{X}(\mathcal{U})} \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\subseteq$}} \\ \hat{S}}}{\hat{\mathbf{X}}(\mathcal{U})} \quad \text{is an isometry.}$$

Example: The helicoid and catenoid are locally isometric.

There are parametrizations $\mathbf{X}, \hat{\mathbf{X}} : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$

Catenoid: $\mathbf{X}(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$

helicoid: $\hat{\mathbf{X}}(u,v) = (\sinh v \cos u, \sinh v \sin u, u)$

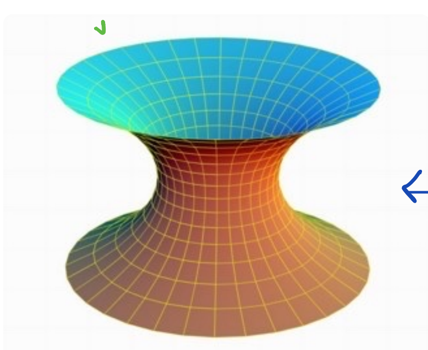
The 1st fundamental form for the catenoid w.r.t. $\mathbf{X}$ is

$$(g_{ij}) = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}$$

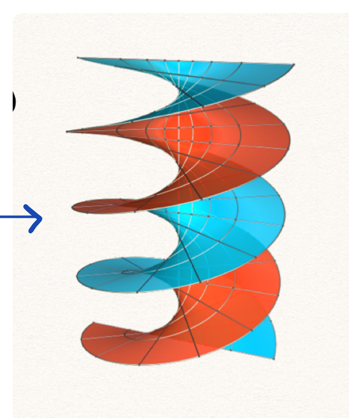
The 1st fundamental form for the helicoid w.r.t. $\hat{\mathbf{X}}$ is

$$(\hat{g}_{ij}) = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}$$

Since $(g_{ij}) = (\hat{g}_{ij})$ on $\mathcal{U} = (0, 2\pi) \times \mathbb{R}$, Prop. above shows that $\hat{\mathbf{X}} \circ \mathbf{X}^{-1}$ is an isometry.



$\xleftrightarrow[\text{isometric}]{\text{locally}}$



§ Calculus of vector fields in $\mathbb{R}^n$

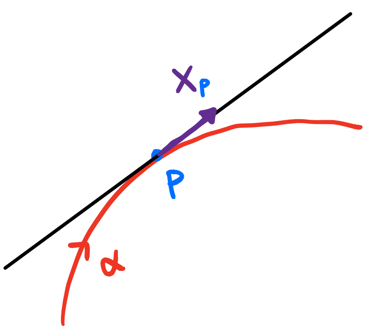
Def$^{\underline{n}}$: (vector fields as directional derivatives)

Given a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$, it defines an operator on smooth functions on $\mathbb{R}^n$:

$$X : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$
$$f \longmapsto X(f)$$

where $X(f)(p) := D_{X_p} f(p)$ directional derivative of f at p along X_p

$$= \frac{d}{dt}\Big|_{t=0} f(p + t X_p)$$



$$= \frac{d}{dt}\Big|_{t=0} f(\alpha(t))$$ for <u>ANY</u> curve α s.t. $\alpha(0) = p$, $\alpha'(0) = X_p$

Properties: (1) Linearity: $X(af + bg) = a\,X(f) + b\,X(g)$

(2) Leibniz rule: $X(fg) = f\,X(g) + g\,X(f)$

where $a, b \in \mathbb{R}$ are constants, $f, g \in C^\infty(\mathbb{R}^n)$.

Note: Given a vector field X and $f \in C^\infty(\mathbb{R}^n)$, one can define a new vector field fX s.t. $(fX)(p) = f(p)\,X(p)$.

Properties: (1) Linearity: $(aX + bY)(f) = a\,X(f) + b\,X(g)$

(2) Tensorial: $(fX)(g) = f(X(g))$

In terms of the Euclidean coordinates $x^1, \dots, x^n$ on $\mathbb{R}^n$,

we can express any vector field $X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as

$$X(x^1, \dots, x^n) = \big(\underbrace{a^1(x^1, \dots, x^n), \dots\dots, a^n(x^1, \dots, x^n)}_{\text{smooth functions}}\big)$$

$$= \sum_{i=1}^{n} a^i e_i \quad , \quad \{e_i\} \text{ std. basis of } \mathbb{R}^n$$

Since e_i corresponds to $\frac{\partial}{\partial x^i}$ as operators, therefore

$$\boxed{X = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + \cdots + a^n \frac{\partial}{\partial x^n}}$$

Since vector fields can be viewed as operators on $C^\infty(\mathbb{R}^n)$,

we can consider their compositions:

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, shift left, "X"] \arrow[r, shift right, "Y"'] & C^\infty(\mathbb{R}^n) \arrow[r, shift left, "Y"] \arrow[r, shift right, "X"'] & C^\infty(\mathbb{R}^n) \end{tikzcd}$$

i.e. $\quad f \longmapsto X(f) \longmapsto Y(X(f))$

or $\quad f \longmapsto Y(f) \longmapsto X(Y(f))$